ASYMPTOTIC STABILITY OF THE SOLUTIONS OF BOUNDARY LAYER EQUATIONS

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The asymptotic behavior of problem of flow of a viscous incompressible fluid past a curved obstacle is investigated in boundary layer approximations.

Conditions under which the solution of this problem reduces to the Blasius solution dealing with a flow past a plate, are explained.

Let us direct the x-axis along the boundary of the obstacle and the y-axis along the normal. We introduce the following notation: u and v are the velocity components along the x- and y-axes, respectively; $U(x)$ is the longitudinal velocity component of the external flow; ν is the viscosity factor. We assume that the density $\rho = 1$. The set of Prandtl equations in domain $P\{0 < x < \infty,\ 0 < y < \infty\}$ and the corresponding boundary conditions expressed in our notation, are of the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial y^2} = U(x)\frac{dU(x)}{dx}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u|_{y=0} = 0, \quad v|_{y=0} = 0, \quad u|_{x=0} = u_0(y), \quad \lim_{y\to\infty} u(x, y) = U(x) \tag{2}$$

where the limit exists and is finite for any value of x, $x \in [0, \infty]$. The function $U(x)$ is related to pressure $p(x)$ by Bernoulli law

$$2p(x) + U^2(x) = C = \text{const}$$

We shall assume that there exists a solution u, v of the set (1) under the conditions (2) in domain P, and that the component $u(x, y)$ has the following properties: when $y > 0$, $u(x, y) > 0$, and $u(x, y)$ is continuous and bounded in domain $P\{0 \leqslant x < \infty,\ 0 \leqslant y < \infty\}$.

The relevant theorem on the existence of a solution for problem (1), (2) was proved in [1] with the assumption that the initial profile $u_0(y)$ and the function $U(x)$ are fairly smooth, and $u_0(y) > 0$ for $y > 0$, $u_0'(0) > 0$,

$$u_0(0) = 0, \quad u_0(\infty) = U(0), \quad dp\,/\,dx \leqslant 0$$

In [2], the behavior of velocity component $u(x, y)$ in the boundary layer (1), (2) in the case of $(dp\,/\,dx \leqslant 0)$ was studied for $x \to \infty$. It was shown that the influence of the initial profile of $u_0(y)$ is small for large x, and that the difference of solutions corresponding to different profiles of $u_0(y)$ tends to zero along the lines of flow for $x \to \infty$.

The author of [3] proves that this difference also tends to zero in the physical plane of variables x and y. In the above mentioned papers it is stated that the velocity profiles $u^1(x, y)$ and $u^2(x, y)$, which correspond to different initial functions $u_0^i(y)$ $(i=1, 2)$, are formed by the action of the same external flow $U(x)$. In the present paper we compare, for large values of x, the velocities $u^1(x, y)$ and $u^2(x, y)$ which correspond not only to different initial functions $u_0^i(y)$ but also to different components $U_i(x)$ $(i = 1, 2)$ of the external flow, in the case when

$$\lim_{x\to\infty} U_i(x) = U_\infty = \text{const} \qquad (i = 1, 2)$$

It is proved that if $\lim_{x\to\infty} |U_1(x) - U_2(x)| = \lim_{y\to\infty} |u_0^1(y) - u_0^2(y)| = 0$

then, for $x \to \infty$, the difference between components $u^1(x, y)$ and $u^2(x, y)$ tends uniformly to zero with respect to y, where $y \in [0, \infty)$. One particular consequence is that the velocity profile $u(x, y)$ which is formed in boundary layer (1), (2) when x is large, converges asymptotically to the well-known Blasius solution [4]

$$u_1 = U_\infty f'(\eta), \qquad \eta = \sqrt{U_\infty}\, y / \sqrt{2\nu(x+1)}$$

which corresponds to the flow past a plate, in the longitudinal direction at the velocity $U(x) \equiv U_\infty$. Let us note that the flow function $f(\eta)$, appearing in Blasius solution, is the solution of the boundary value problem

$$ff'' + f''' = 0, \quad f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1 \tag{3}$$

and represents the monotonically increasing function together with its first derivative.

The estimates characterizing the order of mutual convergence of corresponding solutions, are also obtained.

T h e o r e m. Let the following inequalities be fulfilled:

$$0 \leqslant u_0(y) \leqslant U(0), \quad u_0'(0) > 0, \quad u_0(0) = 0 \tag{4}$$

$$0 \leqslant dU/dx \leqslant M_0/(x+1)^{\gamma_0+1}, \quad \gamma_0 > 0 \tag{5}$$

In this case, if x tends to infinity uniformly with respect to y, $y \in [0, \infty)$, the difference $|u(x, y) - u_1(x, y)| \to 0$, where $u_1(x, y) = U_\infty f'(\eta)$, the $f(\eta)$ is the solution of the boundary value problem (3).

If supplementary inequalities

$$U(0) f'(y - N) \leqslant u_0(y) \qquad y \in [N, \infty) \tag{6}$$

$$|u_0(y) - U(0)| \leqslant M_1 \exp\{-\gamma_1 y^2\} \qquad y \in [0, \infty) \tag{7}$$

are satisfied for some constants N, M_1, and $\gamma_1 > 0$, the following estimate holds:

$$|u(x, y) - u_1(x, y)| \leqslant M/(x+1)^\gamma \tag{8}$$

where M and $0 < \gamma < \gamma_0$ are some constants dependent only on the initial data of the problem.

P r o o f. In Mises' variables

$$x = x, \qquad \psi = \psi(x, y) = \int_0^y u(x, y)\, dy \tag{9}$$

the Prandtl system with respect to the function $\omega = u^2(x, \psi)$ in domain $Q\{0 < x < \infty, 0 < \psi < \infty\}$ reduces to the equation

$$\nu \sqrt{\omega} \frac{\partial^2 \omega}{\partial \psi^2} - \frac{\partial \omega}{\partial x} = -2U(x)\, dU/dx \tag{10}$$

and conditions (2) are transformed into the form

$$\omega|_{\psi=0} = 0, \quad \omega|_{x=0} = \omega_0(\psi), \quad \lim_{\psi\to\infty} \omega(x, \psi) = U^2(x) \tag{11}$$

Let us consider the function

$$\omega_2(x, \psi) = u_2^2(x, \psi) = [U(x) - \delta]^2 [f'(\eta)]^2, \quad \eta = \sqrt{U_\infty}(y - N)/\sqrt{2\nu(x+1)}$$

$$N > 0, \qquad y = \int_0^\psi \frac{d\psi}{u_1}$$

Let δ satisfy the condition $0 < \delta < U(0)$.

As N increases, the function $f'(\eta)$ decreases and

$$\lim_{\psi\to\infty} \omega_2(0,\psi) = [U(0)-\delta]^2 < U^2(0) = \lim_{\psi\to\infty} \omega_0(\psi)$$

so that, by virtue of inequalities (4), for N being fairly large and for all $\psi \geqslant N$, the following inequality is fulfilled:

$$\omega_2(0,\psi) \leqslant \omega_0(\psi) \tag{12}$$

To simplify further discussion let us first assume that $N = 0$ and prove that everywhere in domain Q

$$\omega_2(x,\psi) \leqslant \omega(x,\psi) \tag{13}$$

The difference $z = \omega - \omega_2$ satisfies the equation

$$L(z) \equiv \nu\sqrt{\omega}\,\frac{\partial^2 z}{\partial\psi^2} - \frac{\partial z}{\partial x} + \nu\frac{\partial^2\omega_2}{\partial\psi^2}\,\frac{z}{\sqrt{\omega}+\sqrt{\omega_1}} = F_0(x,\psi)$$

$$F_0(x,\psi) = -2U(x)\frac{dU}{dx}\,[1-(f'(\eta))^2] + \nu\frac{\partial^2\omega_2}{\partial\psi^2}\,\frac{\omega_1-\omega_2}{\sqrt{\omega}+\sqrt{\omega_1}} - 2\delta\frac{dU}{dx}[f'(\eta)]^2$$

$$\omega_1 = u_1^2(\eta) = U_\infty^2\,[f'(\eta)]^2$$

Since

$$|f'(\eta)| \leqslant 1,\ \omega_1-\omega_2 \geqslant 0,\ \frac{\partial^2\omega_2}{\partial\psi^2} \leqslant 0,\ \text{then}\ F_0(x,\psi) \leqslant 0,$$

it may be concluded that inequality $z \geqslant 0$ is valid in domain Q. Indeed, let us assume the opposite, i.e. that $z(x_0,\psi_0) < 0$ at some point $(x_0,\psi_0) \in Q$. The function $z(x,\psi) \geqslant 0$ at the boundary $\Gamma\{x=0,\ \psi=0\}$ of domain Q and

$$\lim_{\psi\to\infty} z(x,\psi) = 2U(x)\,\delta - \delta^2 > 0$$

Hence, the negative minimum of function $z(x,\psi)$ is reached within domain $Q_1\{0 < x \leqslant x_0,\ 0 < \psi < \infty\}$ at some point (x_1,ψ_1). At this point

$$\frac{\partial^2 z}{\partial\psi^2} \geqslant 0, \qquad \frac{\partial z}{\partial x} \leqslant 0, \qquad z < 0$$

We have, therefore, $L(z) > 0$. On the other hand, $L(z) = F_0(x,\psi) \leqslant 0$. Thus, our assumption has led us to a contradiction. Therefore, inequality (13) must be fulfilled everywhere in domain Q. Let

$$u_1(x,\psi) = u_1(\eta)$$

Lemma 1. The estimate

$$|u(x,\psi) - u_1(x,\psi)| \leqslant M_2/(x+1)^{\gamma_2} + \varepsilon_2 \tag{14}$$

is valid in domain Q; in (14) M_2 and $0 < \gamma_2 \leqslant \gamma_0$ are some constants, and $\varepsilon_2 > 0$ is arbitrarily small.

Proof. Difference $\sigma = \omega - \omega_1 + c$ where $c = c(x) = U_\infty^2 - U^2(x)$ satisfies the equation

$$L^1(\sigma) \equiv \nu\sqrt{\omega}\,\frac{\partial^2\sigma}{\partial\psi^2} - \frac{\partial\sigma}{\partial x} + \nu\frac{\partial^2\omega_1}{\partial\psi^2}\,\frac{\sigma}{\sqrt{\omega}+\sqrt{\omega_1}} = \nu\frac{\partial^2\omega_1}{\partial\psi^2}\,\frac{c(x)}{\sqrt{\omega}+\sqrt{\omega_1}}$$

Let us consider an ancillary function

$$W(x,\psi) = \sigma(x,\psi) - \mu(x,\psi) + \varepsilon_1,\quad \mu(x,\psi) = M\exp\{-\alpha\theta(\eta)\}/(x+1)^\gamma$$

For an arbitrary fixed $\varepsilon_1 > 0$, we shall choose the constants M, $\gamma > 0$, $\alpha > 0$, and the function $\theta(\eta)$ in such a manner as to make $W(x,\psi)$ not negative at the boundary Γ of domain Q, and to satisfy the following inequality everywhere in domain Q:

$$L^1(W) \equiv -\nu \frac{\partial^2\omega_1}{\partial\psi^2} \frac{\mu(x,\psi)-c(x)}{\sqrt{\omega}+\sqrt{\omega_1}} - \left[\nu\sqrt{\omega}\frac{\partial^2\mu}{\partial\psi^2} - \frac{\partial\mu}{\partial x}\right] \geqslant 0 \qquad (15)$$

By means of a similar argument as when proving inequality (13), we obtain that $W(x,\psi) \leqslant 0$ in domain Q. For the second term in (15) we have

$$L_1(\mu) = \nu\sqrt{\omega}\frac{\partial^2\mu}{\partial\psi^2} - \frac{\partial\mu}{\partial x} = \frac{U_\infty\alpha\mu(x,\psi)}{2(x+1)}\left\{\frac{\sqrt{\omega}}{\omega_1}\left[\alpha\left(\frac{d\theta}{d\eta}\right)^2 - \frac{d^2\theta}{d\eta^2} + \frac{f''(\eta)}{f'(\eta)}\frac{d\theta}{d\eta}\right] - \frac{f(\eta)}{U_\infty f'(\eta)}\frac{d\theta}{d\eta}\right\} + \frac{\gamma\mu(x,\psi)}{(x+1)}$$

If $\theta_\eta' \geqslant 0$, then

$$L_1(\mu) \leqslant \frac{B(x,\psi)}{f'(\eta)} L_2[\theta(\eta)] + \frac{\gamma\mu(x,\psi)}{(x+1)}, \qquad B(x,\psi) = \frac{\alpha\mu(x,\psi)}{(x+1)}\sqrt{\frac{\omega}{\omega_1}} \qquad (16)$$

$$L_2[\theta(\eta)] = \alpha\left(\frac{d\theta}{d\eta}\right)^2 - \frac{d^2\theta}{d\eta^2} + \left[\frac{f''(\eta)}{f'(\eta)} - k(\eta)f(\eta)\right]\frac{d\theta}{d\eta}$$

Here $k(\eta)$ is an arbitrary function satisfying the condition

$$k(\eta) > 0 \text{ for } \eta > 0, \quad 0 \leqslant k(\eta) \leqslant \sqrt{\omega_1/\omega}$$

Let $\theta(\eta)$ be a twice continuously differentiable function which satisfies the condition

$$\theta_\eta' \geqslant 0, \quad L_2[\theta(\eta)] \leqslant -\delta_0[f'(\eta)], \qquad \delta_0 > 0 \qquad (17)$$

Such function exists for sufficiently small α. Indeed, conditions (17) at some value of δ_0 are satisfied by the function $\theta_1(\eta) = [\omega_1]^{1/4}$ in the interval $[0, \eta_0]$ for sufficiently small η_0 and by the function $\theta_2(\eta) = [f(\eta)]^2 + c_0$ in the interval $[\eta_0, \infty)$ for sufficiently small α.

It is obvious that this function can be constructed by simply extending function $\theta_1(\eta)$ over some interval (η_0, η_1), so that the function thus obtained transforms with the required degree of smoothness into function $\theta_2(\eta)$ when $\eta \geqslant \eta_1$, and satisfies conditions (17) in the interval (η_0, η_1).

Inequality (16) becomes then

$$L_1(\mu) \leqslant \frac{\mu(x,\psi)}{(x+1)}\left[-\frac{\alpha\delta_0}{2}\left(\frac{\omega}{\omega_1}\right)^{1/2} + \gamma\right]$$

For sufficiently small $\gamma < \dfrac{\alpha\delta_0 U(0)}{2U_\infty}$ we have

$$\delta_1 = \frac{\alpha\delta_0 U(0)}{2U_\infty} - \gamma > 0, \qquad L_1(\mu) \leqslant -\frac{\mu(x,\psi)\delta_1}{(x+1)}$$

Hence

$$L^1(W) \geqslant -\nu\frac{\partial^2\omega_1}{\partial\psi^2}\frac{\mu(x,\psi)-c(x)}{\sqrt{\omega}+\sqrt{\omega_1}} + \frac{\delta_1\mu(x,\psi)}{(x+1)}$$

Everywhere in domain Q

$$\frac{\partial^2\omega_1}{\partial\psi^2} = 2\frac{\partial^2 u_1}{\partial y^2}\frac{1}{u_1} \leqslant 0$$

for sufficiently large values $\eta \geqslant \eta_2$, the following estimate holds:

$$\left|\nu\frac{\partial^2\omega_1}{\partial\psi^2}\frac{1}{\sqrt{\omega}+\sqrt{\omega_1}}\right| \leqslant \frac{M_3}{(x+1)}\exp\{-\alpha_0 f^2(\eta)\}, \; \alpha_0 > 0$$

Constants M, γ and α can be therefore chosen so that

$$\mu(x,\psi) - c(x) \geqslant 0 \quad \text{for} \quad \eta \leqslant \eta_2$$

$$\nu\frac{\partial^2\omega_1}{\partial\psi^2}\frac{\mu(x,\psi)-c(x)}{\sqrt{\omega}+\sqrt{\omega_1}} \leqslant \frac{\delta_1\mu(x,\psi)}{(x+1)} \qquad \text{for} \quad \eta \geqslant \eta_2$$

$$\mu(x, \psi) - c(x) \geqslant 0 \quad \text{for } \eta < \eta_2 \qquad \text{(cont.)}$$

Then $L^1(W) \geqslant 0$, and $W(x, \psi) \leqslant 0$ in domain Q.

Allowing for inequality (13) we obtain

$$\omega_2 \equiv u_2^2 \leqslant \omega \leqslant \omega_1 + \mu - c + \varepsilon_1$$

i.e.

$$u_2(x, \psi) \leqslant u(x, \psi) \leqslant \sqrt{\omega_1 + \mu + \varepsilon_1} \leqslant u_1 + \sqrt{\mu} + \sqrt{\varepsilon_1}$$

from which

$$|u(x, \psi) - u_1(x, \psi)| \leqslant |u_1(x, \psi) - u_2(x, \psi)| + \sqrt{\mu(x, \psi)} + \sqrt{\varepsilon_1} \tag{18}$$

However, for some values of constants M, α, γ and δ

$$|u_1(x, \psi) - u_2(x, \psi)| = |U_\infty - U(x) + \delta| f'(\eta) \leqslant \sqrt{\mu(x, \omega)} + \sqrt{\varepsilon_1}$$

Thus, the estimate (14) follows directly from inequality (18).

Let y and y_1 denote the corresponding physical variables of the velocity components $u(x, y)$ and $u_1(x, y)$ for a given value of ψ, i.e.

$$y = \int_0^\psi \frac{d\psi}{u}, \qquad y_1 = \int_0^\psi \frac{d\psi}{u_1}$$

L e m m a 2. For all values of x, $x \in [0, \infty)$, the following inequalities hold:

$$y_1 - y \leqslant M_4 \sqrt{x+1}\, a(x) \ln\left[1 + \frac{bU_\infty}{a(x)}\right] + \frac{a(x)}{bU_\infty} y_1 \tag{19}$$

$$y_1 - y \geqslant \frac{U(x) - U_\infty - \delta}{U(x)} y_1 \tag{20}$$

where

$$a(x) = \frac{M_2}{(x+1)^{\gamma_2}} + \varepsilon_2, \qquad M_4 = [2\nu b^{-1} U_\infty^{-1}]^{1/2}, \qquad b = f'(1)$$

Inequality (19) can be proved by means of the estimate (14), in exactly the same manner as the corresponding estimate was derived in [3] Lemma 4.

Allowing for inequalities $u(x, \psi) \geqslant u_2(x, \psi) \geqslant \dfrac{U(x) - \delta}{U_\infty} u_1(x, \psi)$

we obtain

$$y_1 - y = \int_0^\psi \frac{d\psi}{u_1} - \int_0^\psi \frac{d\psi}{u} \geqslant \int_0^\psi \frac{d\psi}{u_1} - \int_0^\psi \frac{d\psi}{u_2} = \int_0^\psi \frac{d\psi}{u_1} - \frac{U_\infty}{U(x) - \delta} \int_0^\psi \frac{d\psi}{u_1} = \left(1 - \frac{U_\infty}{U(x) - \delta}\right) y_1$$

and this is the proof of estimate (20).

L e m m a 3. The following inequality holds in domain P:

$$|u_1(x, y_1) - u_1(x, y)| \leqslant M_3/(x+1)^{\gamma_3} + \varepsilon_3 \quad (\gamma_3 < \gamma_2) \tag{21}$$

Here $M_3 > M_1$ is some constant, and $\varepsilon_3 \to 0$ when $(\varepsilon_2, \delta) \to 0$.

By virtue of estimate (15) we have

$$y \leqslant y_2 = \int_0^\psi \frac{d\psi}{u_2} = \frac{U_\infty y_1}{U(x) - \delta} \leqslant \frac{U_\infty}{U(0) - \delta} y_1 = y_*$$

Making use of estimates (19) and (20), we obtain from the above

$$|u_1(x, y) - u_1(x, y)| \leqslant \left|\frac{\partial u_1}{\partial y}(x, y_*)\right| |y_1 - y| \leqslant$$

$$\leqslant \left|\frac{\partial u_1}{\partial y}(x, y_*)\right| M_4 \sqrt{x+1}\, a(x) \ln\left[1 + \frac{bU_\infty}{a(x)}\right] + \left|\frac{\partial u_1}{\partial y}(x, y_*)\right| \frac{M_5 a(x)}{bU_\infty + a(x)} y_1$$

Allowing in the estimate of the second term for the inequality

$$\left|\frac{\partial u_1}{\partial y}(x, y_*)\right| = f''\left(\frac{U_\infty}{U(0)-\delta}\eta\right)\left(\frac{U_\infty}{2\nu(x+1)}\right)^{1/2} \leqslant \frac{U_\infty}{y_*}$$

which is valid by virtue of the condition $\partial^2 u_1/\partial y^2 \leqslant 0$, we finally obtain estimate (21).

Let us prove our theorem. We have

$$|u(x, y) - u_1(x, y)| \leqslant |u(x, y) - u_1(x, y_1)| + |u_1(x, y_1) - u_1(x, y)| \leqslant$$
$$\leqslant \frac{M_2}{(x+1)^{\gamma_2}} + \frac{M_3}{(x+1)^{\gamma_3}} + \varepsilon \leqslant \frac{M}{(x+1)^{\gamma}} + \varepsilon \qquad (22)$$

Since ε_1, δ and ε are arbitrary,

$$\lim_{x\to\infty} |u(x, y) - u_1(x, y)| = 0$$

uniformly with respect to y, $y \in [0, \infty)$

If inequalities (6) and (7) are fulfilled, estimate (13) holds when $\delta = 0$, and in determining the function $W(x, \psi)$ we can assume that $\varepsilon_1 = 0$ since α and M were chosen to satisfy conditions $\alpha \leqslant \gamma_1$, $M \geqslant M_1$. We have then $\varepsilon_2 = \varepsilon_3 = \varepsilon = 0$ in inequalities (14), (21) and (22), and estimate (8) follows at once.

In the case of $N \neq 0$ inequality (13) holds only for $\psi \geqslant N$ since function $\omega_2(x, \psi)$ is defined for these values only.

The inequality (18) in Lemma 1 holds only for $\psi \geqslant N$, whereas for $\psi \in [0, N]$ the following inequality is fulfilled :

$$0 \leqslant u(x, \psi) \leqslant |u_1(x, \psi)| + \sqrt{\mu(x, \psi)} + \sqrt{\varepsilon_2}$$

The estimate (14) follows directly from this inequality.

The lower estimate in Lemma 2 is derived in the following manner.

By virtue of the condition $u_0'(0) > 0$, the following constant m_0 $(0 < m_0 < U(0))$ exists for $x = 0$: $m_0 [f'(\eta)]|_{x=0} \leqslant u_0(y)$, $\qquad \eta = \sqrt{m_0}\, y / \sqrt{2\nu(x+1)}$

Hence, on the basis of the principle of the maximum

$$u(x, y) \geqslant m_0 [f'(\eta)] = \frac{m_0}{U_\infty} u_1(\eta)$$

Then

$$y_1 - y \geqslant -2a(x)\frac{U_\infty}{m_0} J \qquad \left(J = \int_0^\psi \frac{d\psi}{u_1(u_1 + a(x))}\right)$$

Having estimated integral J similarly as in Lemma 4 of [3], we finally obtain

$$y_1 - y \geqslant -\frac{2U_\infty M_4}{m_0}\sqrt{x+1}\, a(x) \ln\left[1 + \frac{bU_\infty}{a(x)}\right] + \frac{2a(x) U_\infty}{m_0 (bU_\infty + a(x))} y_1$$

The proof of Lemma 3 and all further arguments are the same as in the case of $N = 0$. The theorem is thus proved.

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